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Deformation of the 'embedding' $A_5 \supset G$

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Abstract. The Cartan–Chevalley generators of G_q, G being a maximal singular sub-algebra of A_5, are written in terms of the generators of $Gl_q(6)$ using a q-boson realization. Then a deformation scheme for A_5 is presented starting from G_q, i.e. in a basis which manifestly exhibits for $q = 1$ the content of the singular subalgebra G.

1. Introduction

The quantum algebras G_q or $U_q(G)$, i.e. the q-deformed universal enveloping algebra of a semisimple Lie algebra G (see, for instance, [1] for a more precise definition), are actually a topic of active research both in physics and mathematics. The underlying idea in some applications of q-algebras is to use a q-deformed algebra instead of a Lie algebra to realize a *generalized dynamical symmetry*. It is well known that the generalized dynamical symmetry in many models of nuclear, hadronic, molecular and chemical physics is displayed through embedding chains of algebras of the type

$$G \supset L \supset \cdots \supset SO(3) \supset SO(2) \tag{1.1}$$

where $SO(3)$ describes the angular momentum and, usually, the Lie algebras are realized in terms of bosonic creation–annihilation operators. In this scheme the Hamiltonian of the system is written as a sum, with constants to be determined from experimental data, of invariants (usually second-order Casimir operators) of the Lie algebras of the chain. An essential step to carry forward the program of application of q-algebras as *generalized dynamical symmetry* is to dispose on a formalism which allows one to build up chains analogous to equation (1.1) replacing the Lie algebras by the deformed ones.

The simplest, not trivial, embedding chain is

$$SU(3) \supset SO(3). \tag{1.2}$$

The $SO(3)$ is the three-dimensional principal subalgebra of $SU(3)$. In [2] the existence of a 3D principal q-subalgebra for $Gl_q(n+1)$ has been investigated, showing that such a subalgebra exists only for $n = 2$ when the algebraic relations are restricted to the symmetric representations, but the coproduct of $Gl_q(3)$ *does not induce* the standard coproduct on the generators of the 3D principal subalgebra. It is useful to emphasize that the definition of the coproduct is essential in order to define the tensor product of spaces.

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In [3] a $SO_q(3)$, i.e. a deformed $SO(3)$ in which the coproduct is imposed in the standard way on the generators, has been defined and a 'deformed $Gl(3)$' has been obtained but, besides some ambiguity in the procedure, the 'deformed $Gl(3)$' is equivalent to the Drinfeld–Jimbo $Gl_q(3)$ as an enveloping algebra, but not as a Hopf algebra. In [4] a deformed $U(3)$ algebra has been constructed in terms of boson operators transforming as a vector under $SO_q(3)$, but also in this approach it is not clear how to endow the 'deformed $U(3)$' algebra with a Hopf structure.

The root of the problem lies in the fact that G_q are well defined only in the Cartan–Chevalley basis and this basis is not suitable to discuss embedding of any subalgebras except trivial ones. Of course, as we are no longer dealing with Lie algebras, the term *embedding* has to be intended in the loose sense that the generators of the embedded deformed subalgebra are expressed in terms of the generators of the algebra, while the Hopf structure can be inherited from that of the embedding algebra or imposed on the generators of the embedded algebra.

In [5] it has been shown that, in the case where the rank of L, the maximal singular algebra of G, is equal to the rank of G minus one, it is possible, using a realization of G_q in terms of q-bosons and/or in terms of the so-called q-fermions, to write the Cartan–Chevalley generators of L_q in terms of the generators of G_q. Let us note that this result is not at all *a priori* obvious due to the nonlinear structure of G_q. The kind of deformed G obtained, if the standard coproduct is imposed on the generators of L_q in the standard way instead of being derived from that of G_q, has also been discussed. However, many problems have been left open, namely the possibility of generalizing the construction to the more general case (the rank of L lower than one with respect to the rank of G) and extending it to any maximal subalgebra J of L. In this paper we address these questions in the case of A_5 which may be of physical interest as the well known Arima–Iachello model is based on this algebra.

In section 2, in order to fix the notation, we recall the definition of the G_q deformation of the universal enveloping algebra of the complex Lie algebra G, in the Chevalley basis, and the definition of q-bosons which we shall use to write explicit realizations of the q-algebras. In section 3 we show that the deformed maximal subalgebras of A_5, i.e. $U_q(A_2)$ and $U_q(D_3)$, can be written in terms of the generators of $Gl_q(6)$ and that this procedure can also be extended to the case of B_2, the maximal subalgebra of $A_4 \subset A_5$. In section 4 we build the deformation of A_5 starting from the deformation of the subalgebra (the L-basis deformation introduced in [5]). In section 5 a few conclusions, remarks and open questions are presented.

2. Reminder of deformed algebras

Let us recall the definition of G_q associated with a simple Lie algebra G of rank r defined by the Cartan matrix (a_{ij}) in the Chevalley basis. G_q is generated by $3r$ elements e_i^+, $f_i = e_i^-$ and h_i which satisfy $(i, j = 1, \ldots, r)$

$$[e_i^+, e_j^-] = \delta_{ij}[h_i]_{q_i} \qquad [h_i, h_j] = 0 \qquad [h_i, e_j^+] = a_{ij}e_j^+ \qquad [h_i, e_j^-] = -a_{ij}e_j^- \tag{2.1}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{2.2}$$

and $q_i = q^{d_i}$, d_i being non-zero integers with their greatest common divisor equal to one such that $d_i a_{ij} = d_j a_{ji}$. Furthermore, the generators have to satisfy the Serre relations

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} (e_i^+)^{1-a_{ij}-n} e_j^+ (e_i^+)^n = 0 \tag{2.3}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q!} \qquad [n]_q! = [1]_q[2]_q \ldots [n]_q. \tag{2.4}$$

Analogous equations hold when e_i^+ is replaced by e_i^-. In the following we assume $h_i = (h_i)^+$ and that the deformation parameter q is different from the roots of unity. The algebra G_q is endowed with a Hopf algebra structure. The action of the coproduct Δ, antipode S and co-unit ε on the generators is as follows:

$$\begin{aligned} &\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \qquad \Delta(e_i^{\pm}) = e_i^{\pm} \otimes q_i^{h_i/2} + q_i^{-h_i/2} \otimes e_i^{\pm} \\ &S(h_i) = -h_i \qquad S(e_i^{\pm}) = -q_i^{\mp 1} e_i^{\pm} \\ &\varepsilon(h_i) = \varepsilon(e_i^{\pm}) = 0 \qquad \varepsilon(1) = 1. \end{aligned} \tag{2.5}$$

As the coproduct in a Hopf algebra satisfies ($g_i, g_j \in G_q$)

$$\Delta(g_i g_j) = \Delta(g_i)\Delta(g_j) \tag{2.6}$$

it is essential to define which elements $\{g_i\}$ are the 'basis' of G_q.

The realization of the q-deformed universal enveloping algebras of the unitary and symplectic series can be obtained [6] as a bilinear of the so-called q-bosons [7]. In the following we will use such a realization, so to fix the notation we recall the definition of q-bosons [7] which we denote by b_i^+ and b_i:

$$b_i b_j^+ - q^{\delta_{ij}} b_j^+ b_i = \delta_{ij} q^{-N_i} \tag{2.7}$$

$$[N_i, b_j^+] = \delta_{ij} b_j^+ \qquad [N_i, b_j] = -\delta_{ij} b_j \qquad [N_i, N_j] = 0. \tag{2.8}$$

It is useful to recall the following identities:

$$b_i^+ b_i = \frac{q^{N_i} - q^{-N_i}}{q - q^{-1}} \qquad b_i b_i^+ = \frac{q^{N_i+1} - q^{-N_i-1}}{q - q^{-1}}. \tag{2.9}$$

It may be useful to stress that, once having realized the generators of the q-algebra G_q as bilinears in the q-bosons, equations (2.1) and (2.3) follow from equations (2.7) and (2.8), but the Hopf structure, equation (2.5), has to be imposed on the generators as a consistent Hopf structure cannot be imposed on the q-bosons. For an explicit construction of q-bosons in terms of non-deformed standard bosonic oscillators see [8]. It turns out that $N_i = \hat{b}_i^+ \hat{b}_i$ where $\hat{b}^+ \hat{b}$ are the non-deformed bosons.

3. Q-embedding chains $U_q(A_5) \supset G_q$

In this section we try to q-deform algebraic chains underlying the interacting boson model (IBM) [9] which is based on the three embedding chains:

$$SU(6) \rightarrow \begin{cases} SU(5) \supset SO(5) \supset SO(3) \supset SO(2) & \text{(I)} \\ SU(3) \supset SO(3) \supset SO(2) & \text{(II)} \\ SO(6) \supset SO(5) \supset SO(3) \supset SO(2) & \text{(III).} \end{cases} \tag{3.1}$$

In this paper we discuss only the q-embedding chains $U_q(A_4) \supset U_q(B_2)$, $U_q(A_5) \supset U_q(A_2)$ and $U_q(A_5) \supset U_q(D_3)$, as $U_q(A_5) \supset U_q(A_4)$ is trivial. We point out that the deformation of the whole chain should require the discussion of the deformation of the real forms of the algebras. See remarks at the end of section 3. In the following constructions the q-algebras are all realized in terms of q-boson operators.

3.1. $U_q(A_4) \supset U_q(B_2)$

The starting point is the q-algebra $U_q(B_2)$ defined through the commutation rules

$$\begin{aligned}[E_i^+, E_j^-] &= \delta_{ij}[H_i]_{q_i} \qquad (i, j = 1, 2)\\ [H_i, E_j^\pm] &= \pm a_{ij} E_j^\pm \end{aligned} \tag{3.2}$$

where a_{ij} is the Cartan matrix

$$a_{ij} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and $q_1 = q^2$, $q_2 = q$, and the q-Serre relations are

$$[E_1^\pm, [E_1^\pm, E_2^\pm]_{q^2}]_{q^{-2}} = 0 \qquad [E_2^\pm, [E_2^\pm, [E_2^\pm, E_1^\pm]_{q^2}]_{q^{-2}}] = 0. \tag{3.3}$$

Let us introduce five q-bosons b_i^+, b_i $(i = 1, \ldots, 5)$, so we can write the generators as

$$H_1 = N_1 - N_2 + N_4 - N_5 \tag{3.4}$$

$$H_2 = 2(N_2 - N_4) \tag{3.5}$$

$$\begin{aligned} E_1^+ = \{&\sqrt{q^{N_1} + q^{-N_1}} b_1^+ b_2 \sqrt{q^{N_2} + q^{-N_2}} q^{-(N_4 - N_5)} \\ &+ \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_5 \sqrt{q^{N_5} + q^{-N_5}} q^{(N_1 - N_2)}\}(q + q^{-1})^{-1} \end{aligned} \tag{3.6}$$

$$\begin{aligned} E_1^- = \{&q^{-(N_4 - N_5)} \sqrt{q^{N_2} + q^{-N_2}} b_2^+ b_1 \sqrt{q^{N_1} + q^{-N_1}} \\ &+ q^{(N_1 - N_2)} \sqrt{q^{N_5} + q^{-N_5}} b_5^+ b_4 \sqrt{q^{N_4} + q^{-N_4}}\}(q + q^{-1})^{-1} \end{aligned} \tag{3.7}$$

$$E_2^+ = q^{N_4} q^{-\frac{1}{2}N_3} \sqrt{q^{N_2} + q^{-N_2}} b_2^+ b_3 + b_3^+ b_4 q^{N_2} q^{-\frac{1}{2}N_3} \sqrt{q^{N_4} + q^{-N_4}} \tag{3.8}$$

$$E_2^- = b_3^+ b_2 q^{N_4} q^{-\frac{1}{2}N_3} \sqrt{q^{N_2} + q^{-N_2}} + q^{N_2} q^{-\frac{1}{2}N_3} \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_3. \tag{3.9}$$

We impose the Hopf structure, equations (2.5), on these elements so obtaining $U_q(B_2)$; note that $b_i^+ b_{i+1}$ $(i = 1, 2, 3, 4)$ are the generators of $U_q(A_4)$ [6], and that the $\sum_i N_i$ commutes with all the generators of $U_q(A_4)$. So we are really writing the generators of $U_q(B_2)$ in terms of $Gl_q(5)$.

3.2. $U_q(A_5) \supset U_q(A_2)$

Also in this case we start from the q-algebra $U_q(A_2)$ defined through the commutation rules

$$\begin{aligned}[E_i^+, E_j^-] &= \delta_{ij}[H_i]_q \qquad (i, j = 1, 2)\\ [H_i, E_j^\pm] &= \pm a_{ij} E_j^\pm \end{aligned} \tag{3.10}$$

where a_{ij} is the Cartan matrix

$$a_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the q-Serre relations are

$$[E_1^\pm, [E_1^\pm, E_2^\pm]_q]_{q^{-1}} = 0 \qquad [E_2^\pm, [E_2^\pm, E_1^\pm]_q]_{q^{-1}} = 0. \tag{3.11}$$

Introducing a set of six q-bosons b_i^+, b_i $(i = 1, \ldots, 6)$ a realization of $U_q(A_2)$ can be written:

$$H_1 = 2N_1 - 2N_4 + N_3 - N_5 \tag{3.12}$$

$$H_2 = N_2 - N_3 + 2N_4 - 2N_6 \tag{3.13}$$

$$E_1^+ = b_3^+ b_5 q^{(N_1-N_4)} + [q^{N_4} q^{-\frac{1}{2}N_2} \sqrt{q^{N_1} + q^{-N_1}} b_1^+ b_2 + b_2^+ b_4 q^{N_1} q^{-\frac{1}{2}N_2} \sqrt{q^{N_4} + q^{-N_4}}] \times q^{-(N_3-N_5)/2} \tag{3.14}$$

$$E_1^- = q^{-(N_3-N_5)/2} [b_2^+ b_1 q^{N_4} q^{-\frac{1}{2}N_2} \sqrt{q^{N_1} + q^{-N_1}} + q^{N_1} q^{-\frac{1}{2}N_2} \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_2] + q^{(N_1-N_4)} b_5^+ b_3 \tag{3.15}$$

$$E_2^+ = b_2^+ b_3 q^{-(N_4-N_6)} + [q^{N_6} q^{-\frac{1}{2}N_5} \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_5 + b_5^+ b_6 q^{N_4} q^{-\frac{1}{2}N_5} \sqrt{q^{N_6} + q^{-N_6}}] \times q^{(N_2-N_3)/2} \tag{3.16}$$

$$E_2^- = q^{(N_2-N_3)/2} [b_5^+ b_4 q^{N_6} q^{-\frac{1}{2}N_5} \sqrt{q^{N_4} + q^{-N_4}} + q^{N_4} q^{-\frac{1}{2}N_5} \sqrt{q^{N_6} + q^{-N_6}} b_6^+ b_5] + q^{-(N_4-N_6)} b_3^+ b_2. \tag{3.17}$$

This $U_q(A_2)$ can be endowed with a Hopf structure in the standard way.

3.3. $U_q(A_5) \supset U_q(D_3)$

The algebra $U_q(D_3)$ is defined by the commutation rules

$$[E_i^+, E_j^-] = \delta_{ij} [H_i]_q \qquad (i, j = 1, 2, 3)$$
$$[H_i, E_j^\pm] = \pm a_{ij} E_j^\pm \tag{3.18}$$

where a_{ij} is the Cartan matrix

$$a_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the q-Serre relations are

$$[E_1^\pm, [E_1^\pm, E_2^\pm]_q]_{q^{-1}} = 0 \qquad [E_2^\pm, [E_2^\pm, E_1^\pm]_q]_{q^{-1}} = 0$$
$$[E_2^\pm, [E_2^\pm, E_3^\pm]_q]_{q^{-1}} = 0 \qquad [E_3^\pm, [E_3^\pm, E_2^\pm]_q]_{q^{-1}} = 0$$
$$[E_1^\pm, E_3^\pm] = 0 \qquad [E_3^\pm, E_1^\pm] = 0. \tag{3.19}$$

Let us now write the generators of $U_q(D_3)$ in terms of six q-bosons b_1^+, $b_i (i = 1, \ldots, 6)$:

$$H_1 = N_2 - N_4 + N_3 - N_5 \tag{3.20}$$

$$H_2 = N_1 - N_2 + N_5 - N_6 \tag{3.21}$$

$$H_3 = N_2 - N_3 + N_4 - N_5 \tag{3.22}$$

$$E_1^+ = b_2^+ b_4 q^{(N_3-N_5)/2} + b_3^+ b_5 q^{-(N_2-N_4)/2} \tag{3.23}$$

$$E_1^- = q^{(N_3-N_5)/2} b_4^+ b_2 + q^{-(N_2-N_4)/2} b_5^+ b_3 \tag{3.24}$$

$$E_2^+ = b_1^+ b_2 q^{(N_5-N_6)/2} + b_5^+ b_6 q^{-(N_1-N_2)/2} \tag{3.25}$$

$$E_2^- = q^{(N_5-N_6)/2} b_2^+ b_1 + q^{-(N_1-N_2)/2} b_6^+ b_5 \tag{3.26}$$

$$E_3^+ = b_2^+ b_3 q^{(N_4-N_5)/2} + b_4^+ b_5 q^{-(N_2-N_3)/2} \tag{3.27}$$

$$E_3^- = q^{(N_4-N_5)/2} b_3^+ b_2 + q^{-(N_2-N_3)/2} b_5^+ b_4. \tag{3.28}$$

Let us impose the Hopf structure on these generators, so we obtain $U_q(D_3)$.

Now let us make a few remarks:

(1) The generators are not invariant for $q \to q^{-1}$ and the change of q with q^{-1} destroys the commutation relations and/or the q-Serre relations.

(2) It is by no means *a priori* evident that the q-Serre relations are satisfied. For instance, in the case of $U_q(A_2)$, we have to compute nine q-commutators of which only two vanish and then compute 18 q^{-1}-commutators of which only nine vanish. Finally, there is a cancellation between the remaining terms.

(3) We have written a realization of $U_q(B_2)$ and $U_q(D_3)$ in terms of q-bosons while in [6] the realization of $U_q(B_n)$ and $U_q(D_n)$ is obtained in terms of the so-called q-fermions.

Further steps of deformation cannot be performed as it is not possible to express $U_q(B_1)$ (i.e. a real form of $U_q(A_1)$ characterized by the property that its representations, for generic q, are of odd dimension) in terms of the generators of $U_q(B_2)$, equations (3.4)–(3.9), or of $U_q(A_2)$, equations (3.12)–(3.17), neither $U_q(B_2)$ in terms of $U_q(D_3)$, equations (3.20)–(3.28). In the last case it is not possible to obtain $q_1 = q^2$ in the defining relations of $U_q(B_2)$.

4. The L-basis for $U_q(A_5)$

We present here an alternative deformation scheme, which has been called in [5] the *L-basis* deformation as it depends on the choice of the subalgebra L, for $U_q(A_5)$, where L is one of the maximal subalgebras of A_5 of section 3. This scheme allows one to discuss 'embedding' chains, in the loose sense explained in section 1, of the type

$$G_q \supset L_q \tag{4.1}$$

L being a maximal subalgebra of G.

We do not present here the general scheme, which has been introduced in [5], but we limit ourselves to recall the main ideas and results, and then to apply them to the cases considered in section 3.

In the case of semisimple Lie algebras, the algebra G can be constructed adding to the subalgebra L a suitable set of elements belonging to the representation, in general reducible, R_L of L, which appears in the decomposition

$$\mathrm{ad}_G \to \mathrm{ad}_L \oplus R_L. \tag{4.2}$$

For a classification and explicit construction of embeddings of semisimple Lie subalgebras see [10], where reference to the pioneering work of Dynkin on the subject can be found. Then it is natural to wonder if an analogue of this procedure can be defined in the case of q-algebras, i.e. to start by L_q and then to add some more suitable generators.

Let us consider the algebra L_q defined in the Chevalley basis, i.e. defined by the set $\{E_i^{\pm}, H_i\}(i = 1, 2, \ldots, l = \text{rank of } L)$ satisfying equations (2.1), (2.3) and (2.5), and written in terms of the Chevalley generators of G_q. Then one cannot invert the procedure, i.e. write the generators of G_q, simply in function of those of L_q, but one has to add some more generators and there is a large ambiguity in the choice of these further elements. In order to reduce the arbitrariness of this choice, we remark that, at our knowledge, in all explicit realizations of the deformed algebras G_q the commuting elements are the same as the elements of the Cartan subalgebra of G. So we impose a *minimal deformation scheme* requiring:

(1) the Cartan subalgebra is *left unmodified* in the deformation;

(2) if the commutator of two generators $g^+, g^- \in G$ gives an element k belonging to the Cartan subalgebra, then the commutator of the corresponding deformed generators gives $[k]_q$.

Then we define a deformation scheme in which the Cartan subalgebra of G, which is partly in the Cartan subalgebra of L, i.e. $\{H_i\}$, and partly in R_L, namely $\{K_j\}$, is left invariant and we add to the generators of L_q the set K_j, whose number is given by the difference between the rank of G and the rank of L and which are chosen in a suitable way, specified below. This deformation scheme will define a deformed algebra, which we denote $\mathcal{G}_q$ in order to distinguish it by the Drinfeld–Jimbo deformed algebra G_q, which clearly *contains* the deformed algebra L_q, i.e. we build the chain

$$\mathcal{G}_q \supset L_q. \tag{4.3}$$

In order to define $\mathcal{G}_q$ we introduce the set K_j such that

$$[K_j, E_i^{\pm}] = \pm X_{j,i}^{\pm} \tag{4.4}$$

$$[H_k, X_{j,i}^{\pm}] = \pm a_{ki} X_{j,i}^{\pm} \qquad [H_i, K_j] = 0 \tag{4.5}$$

$$[X_{j,i}^{+}, X_{j,i}^{-}] = [H_i]_{q_i} \tag{4.6}$$

where $\{E_i^{\pm}, H_i\}$ are the generators of L_q which satisfy equations (2.1), (2.3) and (2.5). $\mathcal{G}_q$ will be defined by the generators of L_q and by the elements $(K_j, X_{j,i}^{\pm})$ which *do not belong* to L_q, so *a priori* no coproduct, antipode or counit is defined on them. We extend the Hopf structure from L_q to $(K_j, X_{j,i}^{\pm})$ as follows:

$$\Delta(K_j) = K_j \otimes 1 + 1 \otimes K_j \qquad \Delta(X_{j,i}^{\pm}) = X_{j,i}^{\pm} \otimes q_i^{H_i/2} + q_i^{-H_i/2} \otimes X_{j,i}^{\pm}$$
$$S(K_j) = -K_j \qquad S(X_{j,i}^{\pm}) = -q_i^{\mp 1} X_{j,i}^{\pm} \qquad \varepsilon(K_j) = \varepsilon(X_{j,i}^{\pm}) = 0. \tag{4.7}$$

Really we have to impose the Hopf structure only on the element K_j; the Hopf structure on $X_{j,i}^{\pm}$ can be derived from equations (2.5) and (2.6), the consistency of the coproduct being ensured by equations (4.4) and (4.6). Let us emphasize once more that $\{H_i, K_j\}$, $(i = 1, \ldots, l)$ are linear combinations of the elements of the basis of the Cartan subalgebra of G which are preserved unmodified in the deformation procedure.

As a result the 'deformed algebra $\mathcal{G}_q$' obtained by this deformation scheme is *not* the usual (Drinfeld–Jimbo) G_q.

The deformation scheme we have just sketched requires that the generators of L_q are expressed as functions of those of G_q. This is by no means evident '*a priori*', but we have shown that it can be really done in the case of $U_q(A_5)$ using explicit constructions in terms of q-bosons.

4.1. $U_q(A_4) \supset U_q(B_2)$

To the generators of $U_q(B_2)$ we add two elements K_j:

$$K_1 = N_1 + N_5 \tag{4.8}$$

$$K_2 = N_3. \tag{4.9}$$

We have

$$[K_1, E_1^{+}] = X_{1,1}^{+}$$
$$X_{1,1}^{+} = \{\sqrt{q^{N_1} + q^{-N_1}}\, b_1^{+} b_2 \sqrt{q^{N_2} + q^{-N_2}}\, q^{-(N_4 - N_5)} - \sqrt{q^{N_4} + q^{-N_4}}\, b_4^{+} b_5 \sqrt{q^{N_5} + q^{-N_5}}\, q^{(N_1 - N_2)}\} (q + q^{-1})^{-1} \tag{4.10}$$
$$[K_1, E_2^{+}] = 0 \qquad [K_2, E_1^{+}] = 0 \qquad [K_2, E_2^{+}] = X_{2,2}^{+}$$

$$X_{2,2}^{+} = -q^{N_4} q^{-\frac{1}{2}N_3} \sqrt{q^{N_2} + q^{-N_2}}\, b_2^{+} b_3 + b_3^{+} b_4 q^{N_2} q^{-\frac{1}{2}N_3} \sqrt{q^{N_4} + q^{-N_4}} \tag{4.11}$$

4.2. $U_q(A_5) \supset U_q(A_2)$

To the generators of $U_q(A_2)$ we add three elements K_j:

$$K_1 = N_1 - N_3 - N_4 + N_6 \tag{4.12}$$
$$K_2 = N_2 + N_3 \tag{4.13}$$
$$K_3 = N_3 + N_5. \tag{4.14}$$

We have

$$[K_1, E_1^+] = X_{1,1}^+$$
$$X_{1,1}^+ = -b_3^+ b_5 q^{(N_1-N_4)} + [q^{N_4} q^{-\frac{1}{2}N_2} \sqrt{q^{N_1} + q^{-N_1}} b_1^+ b_2 + b_2^+ b_4 q^{N_1} q^{-\frac{1}{2}N_2} \sqrt{q^{N_4} + q^{-N_4}}] q^{-(N_3-N_5)/2} \tag{4.15}$$
$$[K_1, E_2^+] = X_{1,2}^+$$
$$X_{1,2}^+ = b_2^+ b_3 q^{-(N_4-N_6)} + [-q^{N_6} q^{-\frac{1}{2}N_5} \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_5 - b_5^+ b_6 q^{N_4} q^{-\frac{1}{2}N_5} \sqrt{q^{N_6} + q^{-N_6}}] q^{(N_2-N_3)/2} \tag{4.16}$$
$$[K_2, E_1^+] = X_{2,1}^+$$
$$X_{2,1}^+ = b_3^+ b_5 q^{(N_1-N_4)} + [-q^{N_4} q^{-\frac{1}{2}N_2} \sqrt{q^{N_1} + q^{-N_1}} b_1^+ b_2 + b_2^+ b_4 q^{N_1} q^{-\frac{1}{2}N_2} \sqrt{q^{N_4} + q^{-N_4}}] q^{-(N_3-N_5)/2} \tag{4.17}$$
$$[K_2, E_2^+] = 0 \qquad [K_3, E_1^+] = 0 \qquad [K_3, E_2^+] = X_{3,2}^+$$
$$X_{3,2}^+ = -b_2^+ b_3 q^{-(N_4-N_6)} + [-q^{N_6} q^{-\frac{1}{2}N_5} \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_5 + b_5^+ b_6 q^{N_4} q^{-\frac{1}{2}N_5} \sqrt{q^{N_6} + q^{-N_6}}] q^{(N_2-N_3)/2}. \tag{4.18}$$

4.3. $U_q(A_5) \supset U_q(D_3)$

To the generators of $U_q(D_3)$ we add two elements K_j:

$$K_1 = N_3 + N_4 \tag{4.19}$$
$$K_2 = N_1 + N_6. \tag{4.20}$$

We have

$$[K_1, E_1^+] = X_{1,1}^+$$
$$X_{1,1}^+ = -b_2^+ b_4 q^{(N_3-N_5)/2} + b_3^+ b_5 q^{-(N_2-N_4)/2} \tag{4.21}$$
$$[K_1, E_3^+] = X_{1,3}^+$$
$$X_{1,3}^+ = -b_2^+ b_3 q^{(N_4-N_5)/2} + b_4^+ b_5 q^{-(N_2-N_3)/2} \tag{4.22}$$
$$[K_2, E_2^+] = X_{2,2}^+$$
$$X_{2,2}^+ = b_1^+ b_2 q^{(N_5-N_6)/2} - b_5^+ b_6 q^{-(N_1-N_2)/2} \tag{4.23}$$
$$[K_1, E_2^+] = 0 \qquad [K_2, E_1^+] = 0 \qquad [K_2, E_3^+] = 0.$$

It is easy to verify that all equations (4.5) and (4.6) are verified. Then we can express the generators of Drinfeld–Jimbo $U_q(A_5)(U_q(A_4))$, see [6], in terms of the generators of $U_q(A_2)$ and $U_q(D_3)(U_q(B_2))$ and of the elements K_j and $X_{j,i}^{\pm}$. Explicitly we find

4.1.1. $U_q(A_4) \supset U_q(B_2)$.

$$e_1^+ = \tfrac{1}{2}(q+q^{-1})(q^{N_1}+q^{-N_1})^{-\frac{1}{2}}(E_1^+ + X_{1,1}^+)(q^{N_2}+q^{-N_2})^{-\frac{1}{2}}q^{(N_4-N_5)} \tag{4.24}$$

$$e_2^+ = \tfrac{1}{2}q^{-N_4}q^{\frac{1}{2}N_3}(q^{N_2}+q^{-N_2})^{-\frac{1}{2}}(E_2^+ - X_{2,2}^+) \tag{4.25}$$

$$e_3^+ = \tfrac{1}{2}(E_2^+ + X_{2,2}^+)q^{-N_2}q^{\frac{1}{2}N_3}(q^{N_4}+q^{-N_4})^{-\frac{1}{2}} \tag{4.26}$$

$$e_4^+ = \tfrac{1}{2}(q+q^{-1})(q^{N_4}+q^{-N_4})^{-\frac{1}{2}}(E_1^+ - X_{1,1}^+)(q^{N_5}+q^{-N_5})^{-\frac{1}{2}}q^{(N_2-N_1)}. \tag{4.27}$$

4.2.1. $U_q(A_5) \supset U_q(A_2)$

$$e_1^+ = \tfrac{1}{2}q^{-N_4}q^{\frac{1}{2}N_2}(q^{N_1}+q^{-N_1})^{-\frac{1}{2}}(E_1^+ - X_{2,1}^+)q^{(N_3-N_5)/2} \tag{4.28}$$

$$e_2^+ = \tfrac{1}{2}(E_2^+ + X_{1,2}^+)q^{(N_4-N_6)} \tag{4.29}$$

$$e_4^+ = -\tfrac{1}{2}q^{-N_6}q^{\frac{1}{2}N_5}(q^{N_4}+q^{-N_4})^{-\frac{1}{2}}(X_{1,2}^+ + X_{3,2}^+)q^{(N_3-N_2)/2} \tag{4.30}$$

$$e_5^+ = \tfrac{1}{2}(E_2^+ + X_{3,2}^+)q^{-N_4}q^{\frac{1}{2}N_5}(q^{N_6}+q^{-N_6})^{-\frac{1}{2}}q^{(N_3-N_2)/2}. \tag{4.31}$$

Now we see that

$$[e_2^+, e_3^+]_q q^{N_3} = \tfrac{1}{2}(X_{1,1}^+ + X_{2,1}^+)q^{-N_1}q^{\frac{1}{2}N_2}(q^{N_4}+q^{-N_4})^{-\frac{1}{2}}q^{(N_3-N_5)/2}.$$

Making the commutator of the above expression with $e_2^- = f_2$, we find

$$[\tfrac{1}{2}(X_{1,1}^+ + X_{2,1}^+)q^{-N_1}q^{\frac{1}{2}N_2}(q^{N_4}+q^{-N_4})^{-\frac{1}{2}}q^{(N_3-N_5)/2}, e_2^-]_q = -q^{(h_2+2)}e_3^+q^{N_3} \tag{4.32}$$

from which we can derive e_3^+.

4.3.1. $U_q(A_5) \supset U_q(D_3)$.

$$e_1^+ = \tfrac{1}{2}(E_2^+ + X_{2,2}^+)q^{(N_6-N_5)/2} \tag{4.33}$$

$$e_2^+ = \tfrac{1}{2}(E_3^+ - X_{1,3}^+)q^{(N_5-N_4)/2} \tag{4.34}$$

$$e_4^+ = \tfrac{1}{2}(E_3^+ + X_{1,3}^+)q^{(N_2-N_3)/2} \tag{4.35}$$

$$e_5^+ = \tfrac{1}{2}(E_2^+ - X_{2,2}^+)q^{(N_1-N_2)/2}. \tag{4.36}$$

Also in this case we have

$$[e_2^+, e_3^+]_q q^{N_3} = \tfrac{1}{2}(E_1^+ - X_{1,1}^+)q^{(N_5-N_3)/2}.$$

Making the commutator of the above expression with e_2^- we can derive e_3^+.

The generators f_i are obtained by Hermitian conjugation (assuming q real) of e_i^+.

5. Conclusions

We have shown that it is possible to write the Cartan–Chevalley generators of $U_q(D_3)$ and $U_q(A_2)$ in terms of the generators of $Gl_q(6)$ and those of $U_q(B_2)$ in terms of $Gl_q(5)$, but that it is not possible to extend further the procedure to obtain $U_q(B_1)$ as was argued in [5]. The realization of the generators of $Gl_q(6)$ in terms of q-bosons has been essential in our derivation. Our approach can be considered as a deformation of the Cartan–Chevalley generators of G, subalgebra of A_5, written in terms of those of A_5, using a boson realization. We emphasize once more that in this way a q-boson realization of orthogonal deformed algebras $U_q(D_3)$ and $U_q(B_2)$ has been obtained. An interesting point to investigate is whether this procedure can be extended to the case of $U_q(D_n)$ and $U_q(B_n)$ for any value of n.

We have then presented a deformation scheme of A_5 in a basis different from the Chevalley basis, where the primitive elements on which the coproduct is imposed in the standard way are the generators of its maximal singular subalgebra and some additive elements which are in the Cartan subalgebra of A_5. In this way a deformed A_5 is obtained in a basis which, in the limit $q = 1$, manifestly exhibits the content of the singular subalgebra G. The obtained deformed algebra A_5 is equivalent, as an enveloping algebra, to the Drinfeld–Jimbo $U_q(A_5)$, defined in the Chevalley basis, but it is endowed by a different Hopf structure. It should be recalled that the R universal matrix for $U_q(A_5)$ is known in the Cartan–Chevalley basis. It is not clear that an analogous R matrix can be defined in our basis. A peculiar feature of this deformation scheme is the fact that the deformed subalgebra $\mathcal{G}_q$ is not always invariant for $q \to q^{-1}$; only the commuting subalgebra is always invariant. Another peculiar feature is the fact that in the L-basis we need the q-Serre relations only on the subalgebra L. Indeed, once the deformed set of generators $\{E_i^{\pm}, H_i; i = 1, \ldots, l\}$, satisfying equations (2.1) and (2.3), are introduced, the properties of the elements $\{K_j, X_{j,i}^{\pm}\}$ are uniquely defined by equations (4.4) and (4.7). However, let us emphasize that we indeed require knowledge of the whole algebra G, while for the deformation in the Chevalley basis only knowledge of the generators corresponding to the simple roots of G is required. We have also shown that the method cannot be applied to any embedding chain as

$$G_q \supset L_q \supset J_q.$$

It is also an interesting problem to study how the representations of $\mathcal{G}_q$ decompose with respect to L_q; in this context the choice of the definition of the coproduct is relevant. Many open problems are still present. In particular, the choice of the set of elements K_j is somewhat arbitrary. In principle, we may generalize the approach presented here to the q-superalgebras.

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